ASYMPTOTICS OF THE HILBERT-SMITH NORM OF CURVE OPERATORS IN TQFT

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ABSTRACT. Applying standard techniques from Toeplitz operator theory, we analyze the asymptotics of the Hilbert-Smith norms of the TQFT operators coming from isotopy classes of one dimensional oriented submanifolds on a closed oriented surface. We thereby obtain a Toeplitz operator interpretation and generalization of the asymptotic formula obtained recently by Marché and Narimannejad in [MN].

1. INTRODUCTION

By the very axioms of a general TQFT (see [T]), we have the following setup.

Any TQFT $\mathcal{Z}$ associates a finite dimensional Hermitian vector space $\mathcal{Z}(\Sigma)$ to a closed oriented surface Σ. It also induces a projective representation

$$\mathcal{Z} : \Gamma \to \mathrm{Aut}(\mathbb{P}(\mathcal{Z}(\Sigma))),$$

where Γ is the mapping class group of Σ.

For any tripel (M, L, λ), where M is compact oriented three manifold, L is a one dimensional oriented framed sub-manifold in $M - \partial M$ and λ is a labeling of each of the components of L by an element from a certain finite label set, which is part of the theory $\mathcal{Z}$, one get a vector

$$\mathcal{Z}(M, L, \lambda) \in \mathcal{Z}(\partial M).$$

What interest us here is the operators associated to labeled simple closed curves. More generally suppose γ is an oriented one dimensional sub-manifold γ of Σ and that λ is a labeling of each of the components of γ. Then by embedding γ in $\Sigma \times [0, 1]$ via the identification of Σ with $\Sigma \times \{1/2\}$ and using the "black board" framing along $\Sigma \times \{1/2\}$, we get the *curve operator*

$$\mathcal{Z}(\gamma, \lambda) \in \mathrm{End}(\mathcal{Z}(\Sigma)).$$

Using the above mentioned Hermitian structure we have the Hilbert-Smith norm or inner product in its polarized form:

$$\langle \mathcal{Z}(\gamma_1, \lambda_1), \mathcal{Z}(\gamma_2, \lambda_2) \rangle = \mathrm{Tr}(\mathcal{Z}(\gamma_1, \lambda_1)\mathcal{Z}(\gamma_2, \lambda_2)^*).$$

In this paper we will compute the large level asymptotics of this Hilbert-Smith Norm for the Reshetikhin-Turaev quantum $SU(n)$ TQFT $Z^{(k)}$ with the use of Toeplitz operator theory. Let us make this precise.

The TQFT's $Z^{(k)}$ was rigorously constructed by Reshetikhin and Turaev in [RT1] and [RT2] using the theory of quantum groups. A purely skein theory model for these TQFT's was developed in [BHMV1], [BHMV2] and [B1].

The label set Λ_k of the theory $Z^{(k)}$ is given as follows: Let Λ be the set of finite dimensional irreducible representations of $SU(n)$. By the theory of dominant

weight modules, Λ is identified with the set of dominant weights of the Lie algebra of $SU(n)$. The label set for $Z^{(k)}$ is the following subset of Λ:

$$\Lambda_k = \{\lambda \in \Lambda | (\lambda, \theta) \le k\},$$

where $(\cdot, \cdot)$ is the Cartan-Killing form and θ is the highest root of the Lie algebra of $SU(n)$ (and $(\cdot, \cdot)$ is normalized by requiring that $(\theta, \theta) = 2$). We observe that any $\lambda \in \Lambda$ is contained in Λ_k for k large enough. The operation of taking the dual representation gives an involution of Λ, which preserves Λ_k for each k.

Let $n \ge 2$ and consider the space $Z^{(k)}(\Sigma)$ associated to a surface of genus $g > 1$. In fact, we will consider a slightly more general situation, namely where we label a point x on Σ by an element from Λ_k. We will only consider the following situation: Let $\lambda_0 \in \Lambda_1$ be the unique element with the property that the exponential map applied to its associated co-weight generate the center of $SU(n)$. Then let $d \in \mathbb{Z}/n\mathbb{Z}$, label x with $d\lambda_0$ and let $Z^{(k)}(\Sigma)$ be the vector space $Z^{(k)}$ associates to Σ label by $d\Lambda_0$ at x. In case $d = 0$, this is simply just the vector space $Z^{(k)}$ associates to the surface Σ without any labeled points.

Suppose now γ is a one dimensional submanifold of Σ and that λ is a labeling of the components of γ by finite dimensional representations. For large enough k, we therefore get the curve operator

$$Z^{(k)}(\gamma, \lambda) \in \operatorname{End}(Z^{(k)}(\Sigma)).$$

If all components of γ are labeled by the defining representation of $SU(n)$, we just write $Z^{(k)}(\gamma)$ for the corresponding curve operator. In section 2 we review the construction of the curve operators $Z^{(k)}(\gamma, \lambda)$.

We now proceed to discuss the semiclassical limit of these sequences of curve operators, namely holonomy functions on the $SU(n)$ moduli space.

Let M be the moduli space of flat $SU(n)$-connections on $\Sigma - x$ with holonomy $d \in \mathbb{Z}/n\mathbb{Z} \cong Z_{SU(n)}$ around x.

Suppose now that each of the components of γ is labeled by a finite dimensional representation of $SU(n)$ as discussed above. Then we have the holonomy function $h_{\gamma,\lambda}$ associated to γ defined on M by taking the product over the components of γ of the trace in the corresponding representation of the holonomy around the component. Note that $h_{\gamma,\lambda}$ only depends on the free homotopy class of γ. Further $h_{\gamma,\lambda}$ is constant if γ is nul-homotopic. If we label all components of γ by the defining representation of $SU(n)$, we simply denote the corresponding function by h_γ.

On the holonomy functions we have the L_2-inner product:

$$\langle h_{\gamma_1, \lambda_1}, h_{\gamma_2, \lambda_2} \rangle = \frac{1}{m!} \int_M h_{\gamma_1, \lambda_1} \overline{h}_{\gamma_2, \lambda_2} \omega^m,$$

where $m = (g - 1)(n^2 - 1)$ and ω is the symplectic form on M.

The main purpose of this paper is to explain how the following theorem can be deduced from standard results from the theory of Toeplitz operators and by the approximation result Theorem 3 proved in [A6].

Theorem 1. *For all pairs of oriented one dimensional submanifolds γ_1, γ_2 of Σ and all labelings λ_1, λ_2 of their components by finite dimensional irreducible representations of $SU(n)$, we have that*

$$\langle h_{\gamma_1, \lambda_1}, h_{\gamma_2, \lambda_2} \rangle = \lim_{k \to \infty} k^{-(g-1)(n^2-1)} \langle Z^{(k)}(\gamma_1, \lambda_1), Z^{(k)}(\gamma_2, \lambda_2) \rangle,$$

To explain how this statement is related to Toeplitz operators, let us review the $SU(n)$ gauge theory construction of $Z^{(k)}(\Sigma)$.

By applying geometric quantization at level k to the moduli space M one gets a vector bundle $\mathcal{H}^{(k)}$ over Teichmüller space $\mathcal{T}$. The fiber of this bundle over a point $\sigma \in \mathcal{T}$ is $\mathcal{H}_\sigma^{(k)} = H^0(M_\sigma, \mathcal{L}_\sigma^k)$, where M_σ is M equipped with a complex structure induced from σ and $\mathcal{L}_\sigma$ is an ample generator of the Picard group of M_σ.

The main result pertaining to this bundle $\mathcal{H}^{(k)}$ is that its projectivization $\mathbb{P}(\mathcal{H}^{(k)})$ supports a natural flat connection. This is a result proved independently by Axelrod, Della Pietra and Witten [ADW] and by Hitchin [H]. Now, since there is an action of the mapping class group Γ of Σ on $\mathcal{H}^{(k)}$ covering its action on $\mathcal{T}$, which preserves the flat connection in $\mathbb{P}(\mathcal{H}^{(k)})$, we get for each k a finite dimensional projective representation, say $Z^{(k)}$, of Γ, namely on the covariant constant sections, $\mathbb{P}(Z^{(k)}(\Sigma))$ of $\mathbb{P}(\mathcal{H}^{(k)})$ over $\mathcal{T}$. This sequence of projective representations $Z^{(k)}$, $k \in \mathbb{N}$ is the *quantum $SU(n)$ representations* of the mapping class group Γ.

For each $f \in C^\infty(M)$ and each point $\sigma \in \mathcal{T}$, we have the *Toeplitz operator*

$$T_{f,\sigma}^{(k)} : H^0(M_\sigma, \mathcal{L}_\sigma^k) \to H^0(M_\sigma, \mathcal{L}_\sigma^k)$$

which is given by

$$T_{f,\sigma}^{(k)} = \pi_\sigma^{(k)}(fs)$$

for all $s \in H^0(M_\sigma, \mathcal{L}_\sigma^k)$, where $\pi_\sigma^{(k)}$ is the orthogonal projection onto $H^0(M_\sigma, \mathcal{L}_\sigma^k)$. We get smooth section of $\mathrm{End}(\mathcal{H}^{(k)})$ over $\mathcal{T}$

$$T_f^{(k)} \in C^\infty(\mathcal{T}, \mathrm{End}(\mathcal{H}^{(k)}))$$

by letting $T_f^{(k)}(\sigma) = T_{f,\sigma}^{(k)}$. See also [A4] and [A5] where we use these operators to prove the asymptotic faithfulness of the quantum $SU(n)$ representations.

The L_2-inner product on $C^\infty(M, \mathcal{L}^k)$ induces an inner product on $H^0(M_\sigma, \mathcal{L}_\sigma^k)$, which in turn induces the operator norm $\| \cdot \|$ on $\mathrm{End}(H^0(M_\sigma, \mathcal{L}_\sigma^k))$. Hence for any $A \in C^\infty(\mathcal{T}, \mathrm{End}(\mathcal{H}^{(k)}))$ we get the smooth function $\|A\|$ on $\mathcal{T}$.

Theorem 2. *For any two smooth functions $f, g \in C^\infty(M)$ one has that*

$$\langle f, g \rangle = \lim_{k \to \infty} k^{-m} \mathrm{Tr}(T_f^{(k)}(T_g^{(k)})^*),$$

where the real dimension of M is $2m$.

We give a proof of this Theorem in section 3. It is a well known result in the field of Toeplitz operators.

As we will see in section 3, Theorem 1 follows from Theorem 2 and the following Theorem:

Theorem 3. *For any one dimensional oriented submanifold γ and any labeling λ of the components of γ, we have that*

$$\lim_{k \to \infty} \|Z^{(k)}(\gamma, \lambda) - T_{h_{\gamma,\lambda}}^{(k)}\| = 0.$$

The proof of this theorem, which we give in [A6], uses only gauge theory and then asymptotic analysis of certain leafwise elliptic operators on the moduli space, plus ideas from conformal field theory. However, it does *not* rely on the our joint work with K. Ueno [AU1], [AU2] and [AU3].

Let us now specialize to the setting discussed by Marché and Narimannejad. The TQFT V_p is the one defined by Blanchet, Habegger, Masbaum and Vogel in

[BHMV1] and [BHMV2], where $p = 2r$, r being an integer and we choose the $2p$'th root of 1 to be $A = -e^{i\pi/2r}$. I.e. this is the Skein theory model for the quantum $SU(2)$ Reshetikhin-Turaev TQFT.

There is also an inner product on $V_p(\Sigma)$ and the representation V_p of Γ on this vector space is unitary.

Let $\mathcal{S}(\Sigma)$ be the free $\mathbb{C}$-vector space generated by isotopy classes of one-dimensional sub-manifolds of Σ. We do not need any orientation in this case since this is the $SU(2)$ theory. The vector space $\mathcal{S}(\Sigma)$ is known to be isomorphic to the vector space of algebraic functions on the $SL(2, \mathbb{C})$ moduli space by the work of Bullock [Bul] and the work of Przytycki and Sikora [PS].

As above we get a sequence of linear maps

$$V_p : \mathcal{S}(\Sigma) \rightarrow \mathrm{End}(V_p(\Sigma)),$$

and the inner product on $V_p(\Sigma)$ induces an inner product $\langle \cdot, \cdot \rangle_p$ on $\mathrm{End}(V_p(\Sigma))$.

In [MN] Marché and Narimannejad proved

Theorem 4 (Marché and Narimannejad). *For all $\gamma_1, \gamma_2 \in \mathcal{S}(\Sigma)$*

$$\langle h_{\gamma_1}, h_{\gamma_2} \rangle = \lim_{p \to \infty} 2^{(3g-3)} p^{-(3g-3)} \langle V_p(\gamma_1), V_p(\gamma_2) \rangle_p,$$

where M is the moduli space of flat $SU(2)$-connections, i.e. $(n, d) = (2, 0)$.

We have in this paper interpreted this formula in terms of Toeplitz operators and in the process generalize the formula to the $SU(n)$-case. In fact we have given an alternative proof of their theorem, since we can deduce it from Theorem 1:

Our joint work with K. Ueno [AU1], [AU2] and [AU3], combined with the work of Laszlo [La1] gives the following result for the $(n, d) = (2, 0)$ theory:

Theorem 5 (AU). *There is a projective linear isomorphism of representations of Γ*

$$I_k : \mathbb{P}(V_{2k+4}(\Sigma)) \rightarrow \mathbb{P}(Z^{(k)}(\Sigma)).$$

The projective linear isomorphism I_k induces a linear algebra isomorphism of representations of Γ

$$I_k^e : \mathrm{End}(V_{2k+4}(\Sigma)) \rightarrow \mathrm{End}(Z^{(k)}(\Sigma)).$$

From the construction of I_k it follows that

Theorem 6. *For any one dimensional submanifold γ of Σ we have that*

$$I_k^e(V_{2k+4}(\gamma)) = Z^{(k)}(\gamma).$$

We see that Theorem 6 and Theorem 1 implies Marché and Narimannejad's Theorem 4: Using the the algebra isomorphism I_k^e, we compute that

$$
\begin{aligned}
\mathrm{Tr}(V_{2k+4}(\gamma_1) V_{2k+4}(\gamma_2)) &= \mathrm{Tr}(I_k^e(V_{2k+4}(\gamma_1) V_{2k+4}(\gamma_2))) \\
&= \mathrm{Tr}(I_k^e(V_{2k+4}(\gamma_1)) I_k^e(V_{2k+4}(\gamma_2))) \\
&= \mathrm{Tr}(Z^{(k)}(\gamma_1) Z^{(k)}(\gamma_2)).
\end{aligned}
$$

Now we apply Theorem 1 to get Theorem 4 as a special case, since the curve operators $V_{2k+4}(\gamma)$ are self-adjoint.

This paper is organized as follows. In section 2 we review the construction of the curve operators $Z(\gamma, \lambda)$, and in section 3 we prove theorem 1.

We would like to thank Gregor Masbaum for discussions related to this work.

2. Construction of the curve operators

In this section we review how the curve operators $Z^{(k)}(\gamma, \lambda)$ are constructed for the theory $Z^{(k)}$.

Let Σ' be the surface with boundary obtained from cutting Σ along γ. Then any labeling μ of γ by elements from Λ_k induces a labeling of the boundary components of Σ', using the following convention: If Σ' induces the same orientation on a component of $\partial\Sigma'$ as γ does, we use the label for that component, else we use the dual of the label. We also denote this labeling of $\partial\Sigma'$ by μ.

Since $Z^{(k)}$ is also a modular functor one can factor the space $Z^{(k)}(\Sigma)$ into a direct sum "along" γ as a sum over all labelings of γ. That is we get an isomorphism

$$(1) \qquad\qquad Z^{(k)}(\Sigma) \cong \bigoplus_\mu Z^{(k)}(\Sigma', \mu).$$

The sum here runs over all labelings μ of γ by elements from Λ_k. Strictly speaking we need base points on all the components of $\partial\Sigma'$ to define $Z^{(k)}(\Sigma', \mu)$. However, the corresponding subspaces of $Z^{(k)}(\Sigma)$ does not depend on the choice of base points. The isomorphism (1) induces an isomorphism

$$\operatorname{End}(Z^{(k)}(\Sigma)) \cong \bigoplus_\mu \operatorname{End}(Z^{(k)}(\Sigma', \mu)).$$

which also induces a direct sum decomposition of $\operatorname{End}(Z^{(k)}(\Sigma))$ which is independent of the choice of the base points.

The curve operator $Z^{(k)}(\gamma, \lambda)$ is diagonal with respect to this direct sum decomposition along γ. One has the formula

$$Z^{(k)}(\gamma, \lambda) = \bigoplus_\mu S_{\lambda,\mu}(S_{0,\mu})^{-1} \operatorname{Id}_{Z^{(k)}(\Sigma',\mu)}.$$

Here

$$S_{\lambda,\mu} = \prod_{i=1}^{s} S_{\lambda(\gamma_i),\mu(\gamma_i)}$$

where γ_i, $i = 1,\ldots,s$ are the components of γ, and S_{λ_1,λ_2}, $\lambda_1,\lambda_2 \in \Lambda_k$ is the S-matrix of the theory $Z^{(k)}$. See e.g. [B1] for a derivation of this.

In the gauge theory picture, the decomposition (1) is obtained as follows (see [A6] for the details):

One considers a one parameter family of complex structures $\sigma_t \in \mathcal{T}$, $t \in \mathbb{R}_+$, such that the corresponding family in the moduli space of curves converges in the Mumford-Deligne boundary to a nodal curve, which topologically corresponds to shrinking γ to a point. By the results of [A2] the corresponding sequence of complex structures on the moduli space M converges to a non-negative polarization on M, whose isotropic foliation is spanned by the Hamiltonian vector fields associated to the holonomy functions of γ. The main result of [A6] is that the covariant constant sections of $\mathcal{H}^{(k)}$ along the family σ_t converges to distributions supported on the Bohr-Sommerfeld leaves of the limiting non-negative polarization as t goes to infinity. The direct sum of the geometric quantization of the level k Bohr-Sommerfeld levels of the limit non-negative polarization is precisely the left hand side of (1). A sewing-construction inspired from conformal field theory (see [TUY])

is then applied to show that the resulting linear map from the right hand side of
(1) to the left hand side is an isomorphism. This is described in detail in [A6].

Theorem 3 follows from this asymptotic study: One considers the explicit expression for the S-matrix, as given in formula (13.8.9) in Kac's book [Kac]

$$(2) \qquad\qquad S_{\lambda,\mu}/S_{0,\mu} = \lambda(e^{-2\pi i \frac{\check{\mu}+\check{\rho}}{k+n}}),$$

where ρ is half of the sum of the positive roots and $\check{\nu}$ (ν any element of Λ) is the
unique element of the Cartan subalgebra of the Lie algebra of $SU(n)$ which is dual
to ν with respect to the Cartan-Killing form $(\cdot, \cdot)$.

From the expression (2) one sees that under the isomorphism $\check{\mu} \mapsto \mu$, $S_{\lambda,\mu}/S_{0,\mu}$
makes sense for any $\check{\mu}$ in the Cartan subalgebra of the Lie algebra of $SU(n)$. Furthermore one finds that the values of this sequence of functions (depending on k)
is asymptotic to the values of the holonomy function $h_{\gamma,\lambda}$ at the level k Bohr-
Sommerfeld sets of the limiting non-negative polarizations discussed above (see
[A1]). From this one can deduce Theorem 3. See again [A6] for details.

3. Toeplitz operators and proof of Theorem 1.

Let us recall basics on the theory of Toeplitz operators and apply this theory to
prove Theorem 1.

On $C^\infty(M, \mathcal{L}^k)$ we have the L_2-inner product:

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_M (s_1, s_2) \omega^m$$

where $s_1, s_2 \in C^\infty(M, \mathcal{L}^k)$ and $(\cdot, \cdot)$ is the fiberwise Hermitian structure in $\mathcal{L}^k$.

Now let $\sigma \in \mathcal{T}$. Then this L_2-inner product gives the orthogonal projection

$$\pi_\sigma^{(k)} : C^\infty(M, \mathcal{L}^k) \to H^0(M_\sigma, \mathcal{L}_\sigma^k).$$

For each $f \in C^\infty(M)$ consider the associated *Toeplitz operator* $T_{f,\sigma}^{(k)}$ given as the
composition of the multiplication operator

$$M_f : H^0(M_\sigma, \mathcal{L}_\sigma^k) \to C^\infty(M, \mathcal{L}^k)$$

with the orthogonal projection:

$$T_{f,\sigma}^{(k)}(s) = \pi_\sigma^{(k)}(fs).$$

Then $T_{f,\sigma}^{(k)} \in \text{End}(H^0(M_\sigma, \mathcal{L}_\sigma^k))$, and we get a smooth section

$$T_f^{(k)} \in C^\infty(\mathcal{T}, \text{End}(\mathcal{H}^{(k)}))$$

by letting $T_f^{(k)}(\sigma) = T_{f,\sigma}^{(k)}$ (see [A4]).

The L_2-inner product on $C^\infty(M, \mathcal{L}^k)$ induces an inner product on $H^0(M_\sigma, \mathcal{L}_\sigma^k)$,
which in turn induces the operator norm $\| \cdot \|$ on $\text{End}(H^0(M_\sigma, \mathcal{L}_\sigma^k))$.

We need the following Theorem on Toeplitz operators due to Bordemann, Meinrenken and Schlichenmaier (see [BMS], [Sch], [Sch1] and [Sch2]).

Theorem 7 (Bordemann, Meinrenken and Schlichenmaier). *For any $f \in C^\infty(M)$
we have that*

$$\lim_{k \to \infty} \|T_{f,\sigma}^{(k)}\| = \sup_{x \in M} |f(x)|.$$

Since the association of the sequence of Toeplitz operators $T_{f,\sigma}^k$, $k \in \mathbb{Z}_+$ is linear in f, we see from this Theorem, that this association is faithful.

We also have the following two theorems from [BMS]:

Theorem 8 (Bordemann, Meinrenken and Schlichenmaier). *For any $f, g \in C^\infty(M)$ we have that*

$$\|T_{f,\sigma}^{(k)} T_{g,\sigma}^{(k)} - T_{fg,\sigma}^{(k)}\| = O(k^{-1}).$$

Theorem 9 (Bordemann, Meinrenken and Schlichenmaier). *For any $f \in C^\infty(M)$ we have that*

$$\lim_{k \to \infty} k^{-m} \operatorname{Tr}(T_{f,\sigma}^{(k)}) = \frac{1}{m!} \int_M f\omega^m.$$

Proof of Theorem 2. It is immediate from the definition of a Toeplitz operator that

$$(T_{f,\sigma}^{(k)})^* = T_{\bar{f},\sigma}^{(k)}.$$

But then by Theorem 9 and 8

$$
\begin{aligned}
\langle f, g \rangle &= \lim_{k \to \infty} k^{-m} \operatorname{Tr}(T_{f\bar{g},\sigma}^{(k)}) \\
&= \lim_{k \to \infty} k^{-m} \operatorname{Tr}(T_{f,\sigma}^{(k)} T_{\bar{g},\sigma}^{(k)}) \\
&= \lim_{k \to \infty} k^{-m} \operatorname{Tr}(T_{f,\sigma}^{(k)} (T_{g,\sigma}^{(k)})^*).
\end{aligned}
$$

$\square$

Using Theorem 3 we can now give a Toeplitz operator proof of Theorem 1.

Proof of Theorem 1. By Theorem 3 we get that

$$\lim_{k \to \infty} \|T_{h_{\gamma_1},\lambda_1}^{(k)} T_{h_{\gamma_2},\lambda_2}^{(k)} - Z^{(k)}(\gamma_1, \lambda_1) Z^{(k)}(\gamma_2, \lambda_2)\| = 0.$$

But now recall that the Hilbert-Smith norm is bounded by the square root of the dimension of $H^0(M_\sigma, \mathcal{L}_\sigma^k)$ times the operator norm. Since the dimension of $H^0(M_\sigma, \mathcal{L}_\sigma^k)$ grows like k^m, we certainly get that

$$\lim_{k \to \infty} k^{-m} |\operatorname{Tr}(T_{h_{\gamma_1}}^{(k)} (T_{h_{\gamma_2}}^{(k)})^*) - \operatorname{Tr}(Z^{(k)}(\gamma_1, \lambda_1)(Z^{(k)}(\gamma_2, \lambda_2))^*)| = 0.$$

But now the desired result follows from Theorem 2.

$\square$

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